

LIFT OF FROBENIUS AND DESCENT TO CONSTANTS

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ABSTRACT. In differential algebra, a proper scheme X defined over an algebraically closed field K with a derivation ∂ on it descends to the field of constants K^∂ if X itself lifts the derivation ∂ . This is a result by A. Buium. Now in the arithmetic case, the notion of a derivation is replaced by the notion of a π -derivation δ or equivalently in the flat case, a lift of Frobenius ϕ . We will show an analogous result in the arithmetic case of equal characteristic. We show our results using the arithmetic analogue of Taylor expansion using Witt vectors.

1. INTRODUCTION

In [Bui87], Buium shows that given a proper smooth scheme X over an algebraically closed field K with a derivation ∂ on it, if the structure sheaf \mathcal{O}_X has a derivation lifting the one on K , then X descends to the field of constants $K^\partial := \{x \in K \mid \partial x = 0\}$. Consequently, Gillet gave another proof of this result [G02]. In this paper, we will prove an analogous result in the arithmetic case where the notion of derivation is replaced by a π -derivation δ or equivalently in the flat case, by a lift of Frobenius ϕ . Before we state our main results, we would like to give a brief background.

By a tuple of (B, \mathfrak{p}) we will understand the following data: B is a Dedekind domain, \mathfrak{p} a fixed non-zero prime of B with $k := B/\mathfrak{p}$ a finite field and let $q = |k|$. Then the identity map $\mathbb{1} : B \rightarrow B$ is a q -power lift of Frobenius since for all $x \in B$, $x \equiv x^q \pmod{\mathfrak{p}}$. Let R be the \mathfrak{p} -adic completion of B and $\iota : B \rightarrow R$ be the canonical injective map. Also let $\pi \in B$ be such that $\iota(\pi)$ is a generator of the maximal ideal in $\mathfrak{m} \subset R$. Since ι is an injection, we will sometimes, by abuse of notation, consider π as an element of R as well. We will call B or R is of equal characteristic if $\text{char } B = \text{char } R = \text{char } k$. Let $S = \text{Spec } B$.

Let A be a R -algebra. Then as in [Bor11a] one can define the π -typical Witt vectors $W(A)$ with respect to R . For example, when $R = \mathbb{Z}_p$ and $\mathfrak{m} = (p)$ for some prime p , then $W(A)$ are the usual p -typical Witt vectors.

Let A be a flat separated π -adically complete R -algebra with a q -power Frobenius ϕ which is identity on R . Then one can consider an operator called the π -derivation δ on A associated to ϕ given by $\delta x = \frac{\phi(x) - x^q}{\pi}$ for all $x \in A$. By the universal property of Witt vectors, we obtain a canonical map $\exp_\delta : A \rightarrow W(A)$ given by

$$(1.1) \quad \exp_\delta(x) = (P_0(x), P_1(x), \dots)$$

where $P_0(x) = x$, $P_1(x) = \delta x$. This map should be viewed as the analogue of the *Hasse-Schmidt* map, $\exp_\partial : A \rightarrow D(A) := A[[t]]$ in the case of usual derivation ∂ in

differential algebra, given by

$$(1.2) \quad \exp_{\partial}(x) = \sum_{i=0}^{\infty} \frac{\partial^{(i)}x}{i!}$$

Let $A_0 := A/\pi A$ and u be the quotient map $u : A \rightarrow A_0$. Then consider the composition $\overline{\exp}_{\delta} : A \rightarrow W(A_0)$ given by $\overline{\exp}_{\delta} := W(u) \circ \exp_{\delta}$. We say that $\overline{\exp}_{\delta}$ is the *arithmetic Taylor expansion centered at (π)* and is an arithmetic analogue of the usual Taylor expansion principle—this is discussed in detail in section 3.

Analogous to differential algebra, we define the set of δ -constants of A as $A^{\delta} := \{x \in A \mid \delta x = 0\}$. Then it is easy to see that it is a multiplicatively closed set. However if R is of equal characteristic, then A^{δ} is also closed under addition and A^{δ} is then a subring of A . Our first result, theorem 1.1, states that the arithmetic Taylor expansion $\overline{\exp}_{\delta}$ is injective.

Theorem 1.1. *Let A be a separated flat R -algebra with a π -derivation δ (equivalently a q -power Frobenius ϕ) on it. Then the arithmetic Taylor expansion map $\overline{\exp}_{\delta} : A \rightarrow W(A_0)$ is an injection.*

As a consequence we obtain corollary 4.8 which shows that the π -adic topology on such an A with a π -derivation δ on it, is obtained by the restriction of the topology on $W(A_0)$ induced from the system of open neighbourhoods $\{I_n\}_n$ around 0, where for each $n \geq 1$,

$$I_n := \{x \in W(A_0) \mid x = (x_0, x_1, \dots), x_i \in A_0 \text{ and } x_0 = x_1 = \dots = x_{n-1} = 0\}$$

Theorem 1.2. *Assume that R is of equal characteristic. Let A be a separated flat π -adically complete R -algebra and with a lift of Frobenius ϕ which induces an isomorphism on A_0 . Then $A^{\delta} \simeq A_0$.*

Furthermore we have $A \simeq A^{\delta}[[\pi]] \simeq A_0[[\pi]]$.

In the above result, we would like to remark that it follows from the theory of Witt vectors that such an A is isomorphic to $W(A_0) \simeq A_0[[\pi]]$. The key point of the result is to show that the subring of constants A^{δ} is isomorphic to A_0 .

Assume further that B is of equal characteristic and is also a k -algebra where recall $k = B/\mathfrak{p}$. Let X be a scheme over S such that its structure sheaf \mathcal{O}_X has a lift of q -power Frobenius ϕ which restricts to identity on \mathcal{O}_S . Let $\overline{X} = X \times_S (\text{Spec } k)$. Furthermore, we will say ϕ is perfect if ϕ induces an automorphism on $\mathcal{O}_{\overline{X}}$.

Then as an application of Theorem 1.2 we prove the following analogous statement to the result on descending to constants in differential algebra [Bui87].

Theorem 1.3. *Let $S = \text{Spec } B$ and X be a proper scheme flat over S with a q -power lift of Frobenius ϕ on \mathcal{O}_X which is also perfect. Then there exists an étale neighborhood $S' \rightarrow S$ of $\mathfrak{p} \in S$ such that if $X' = X \times_S S'$, then $X' \simeq X_0 \times_{\text{Spec } k} S'$ where $X_0 = X \times_S \text{Spec } k$ is the closed fiber of X at \mathfrak{p} .*

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2. REVIEW OF WITT VECTORS

Witt vectors over a general Dedekind domain with finite residue fields was developed in [Bor11a]. For the sake of our article, we will briefly review the general construction. Let B be a Dedekind domain and fix a maximal ideal $\mathfrak{p} \in \text{Spec } B$ with $k := B/\mathfrak{p}$ a finite field and let $q = |k|$. Let R be the \mathfrak{p} -adic completion of B . Denote by \mathfrak{m} the maximal ideal of the complete, local ring R and $\iota : B \hookrightarrow R$ the natural inclusion. Then let $\pi \in B$ be such that $\iota(\pi)$ generates the maximal ideal \mathfrak{m} in R . Since ι is an injection, by abuse of notation, we will consider π as an element of B as well. Then $k \simeq R/(\pi)$. Do note here that the identity map on R lifts the q -power Frobenius on $R/(\pi)$. We will now review the theory of π -typical Witt vectors over R with maximal ideal \mathfrak{m} . All the rings in this section are R -algebras.

Let C be an A -algebra with structure map $u : A \rightarrow C$. In this paper, any ring homomorphism $\psi : A \rightarrow C$ will be called the *lift of Frobenius* if it satisfies the following:

- (1) The reduction mod π of ψ is the q -power Frobenius, that is, $\psi(x) \equiv u(x)^q \pmod{\pi C}$.
- (2) The restriction of ψ to R is identity.

Let C be an A -algebra with structure map $u : A \rightarrow C$. A π -derivation δ from A to C means a set theoretic map satisfying the following for all $x, y \in B$

$$\begin{aligned} \delta(x + y) &= \delta(x) + \delta(y) + C_\pi(u(x), u(y)) \\ \delta(xy) &= u(x)^q \delta(y) + u(y)^q \delta(x) + \pi \delta(x) \delta(y) \end{aligned}$$

such that δ when restricted to R is $\delta(r) = (r - r^q)/\pi$ for all $r \in R$ and

$$C_\pi(X, Y) = \begin{cases} 0, & \text{if } R \text{ is positive characteristic} \\ \frac{X^{q+Y^q} - (X+Y)^q}{\pi}, & \text{otherwise} \end{cases}$$

It follows that the map $\phi : A \rightarrow C$ defined as

$$\phi(x) := u(x)^q + \pi \delta(x)$$

is an \hat{A} -algebra homomorphism and is a lift of the Frobenius. On R , the π -derivation δ associated to ϕ is given by $\delta x = \frac{\phi(x) - x^q}{\pi}$. Considering this operator δ leads to Buium's theory of arithmetic jet spaces [Bui95, Bui00, Bui09].

Note that this definition depends on the choice of uniformizer π , but in a transparent way: if π' is another uniformizer, then $\delta(x)\pi/\pi'$ is a π' -derivation, and this correspondence induces a bijection between π -derivations and π' -derivations.

We will present three different but equivalent point of views of Witt vectors:

- (1) Given an R -algebra A , the ring of π -typical Witt vectors $W(A)$ can be defined as the unique R -algebra $W(A)$ with a π -derivation δ on $W(A)$ such that, given any R -algebra C with a π -derivation δ on it and an R -algebra map $f : C \rightarrow A$, there exists a unique R -algebra homomorphism $g : C \rightarrow$

$W(A)$ satisfying-

$$\begin{array}{ccc} & W(B) & \\ \downarrow & \swarrow g & \\ A & \xleftarrow{f} & C \end{array}$$

and g satisfies $g \circ \delta = \delta \circ g$. In [Bor11a] (following the approach of [J85] to the usual p -typical Witt vectors), the existence of such a $W(A)$ is shown and that it is also obtained from the classical definition of Witt vectors using ghost vectors.

- (2) However, if we only restrict to flat R -algebras, then the Witt vectors may be classified with the universal property of the lift of Frobenius as follows- given a flat R -algebra A , the ring of π -typical Witt vectors $W(A)$ can be defined as the unique flat R -algebra $W(A)$ with a lift of q -power Frobenius $F : W(A) \rightarrow W(A)$ on it such that, given any flat R -algebra C with a lift of q -power Frobenius ϕ on it and a R -algebra map $f : C \rightarrow A$, there exists an unique R -algebra homomorphism $g : C \rightarrow W(A)$ satisfying-

$$\begin{array}{ccc} & W(A) & \\ \downarrow & \swarrow g & \\ A & \xleftarrow{f} & C \end{array}$$

and g satisfies $g \circ \phi = F \circ g$.

- (3) Here we will review the ghost vector definition of Witt vectors.

For any R -algebra A , define the $n+1$ -fold product as $\Pi_n A = A \times \cdots \times A$ and the infinite product $\Pi_\infty A = A \times A \times \cdots$. Then for all $n \geq 1$ there exists an R -algebra map $T_w : \Pi_n A \rightarrow \Pi_{n-1} A$ given by $T_w(w_0, \dots, w_n) = (w_0, \dots, w_{n-1})$. For all $n \geq 1$ define the left shift operator $F_w : \Pi_n A \rightarrow \Pi_{n-1} A$ as $F_w(w_0, \dots, w_n) = (w_1, \dots, w_n)$

A priori consider the following just as a product of sets $W_n(A) := A^{n+1}$ and the map $w : W_n(A) \rightarrow \Pi_n A$ given by $w(x_0, \dots, x_n) = (w_0, \dots, w_n)$ where

$$(2.1) \quad w_i = x_0^{q^i} + \pi x_1^{q^{i-1}} + \cdots + \pi^i x_i.$$

The map w is known as the *ghost* map. We define the \mathfrak{p} -typical (or π -typical) Witt vectors $W_n(A)$ by the following theorem

Theorem 2.1. *For each $n \geq 0$, there exists a unique ring structure on $W_n(A)$ such that w becomes a natural transformation of functors of rings.*

The proof of this theorem is similar to the p -typical case showed in [H05].

Now we recall some important maps of the Witt vectors. Let $W_n(A)$ denote the truncated Witt vectors of length $n+1$. Then for every $n \geq 1$ there is a *restriction* map $T : W_n(A) \rightarrow W_{n-1}(A)$ given by $T(x_0, \dots, x_n) = (x_0, \dots, x_{n-1})$. Note that T makes $\{W_n(A)\}$ an inverse system and we have $W(A) = \lim_{\leftarrow} W_n(A)$.

For every $n \geq 1$, the *Frobenius* ring homomorphism $F : W_n(A) \rightarrow W_{n-1}(A)$ can be described in terms of the ghost vector. The Frobenius $F : W_n(A) \rightarrow W_{n-1}(A)$ is the unique map that makes the following diagram commutative in a functorial

way

$$(2.2) \quad \begin{array}{ccc} W_n(A) & \xrightarrow{w} & \Pi_n A \\ F \downarrow & & \downarrow F_w \\ W_{n-1}(A) & \xrightarrow{w} & \Pi_{n-1} A^n \end{array}$$

For all $n \geq 0$, we have the multiplicative Teichmüller map $\theta : A \rightarrow W_n(A)$ given by $x \mapsto (x, 0, 0, \dots)$. If A is of positive characteristic, then θ is also additive and hence θ becomes an \mathbb{F}_q -algebra homomorphism.

Given an R -algebra C with a π -derivation δ on it and a $f : C \rightarrow A$, we will now describe the universal map $g : C \rightarrow W(A)$.

It is enough to show in the case when both A and C are flat over R . In that case the ghost map $w : W(A) \rightarrow \Pi_\infty A$ is injective. Consider the map $[\phi] : C \rightarrow \Pi_\infty C$ given by $x \mapsto (x, \phi(x), \phi^2(x), \dots)$. Let for any R -algebra D , $F_w : \Pi_\infty D \rightarrow \Pi_\infty D$ be the left shift operator, defined by $F_w(d_0, d_1, \dots) = (d_1, d_2, \dots)$.

$$\begin{array}{ccccc} & & & C & \\ & & g \nearrow & \downarrow [\phi] & \\ W(A) & \xrightarrow{w} & \Pi_\infty A & \xleftarrow{f} & \Pi_\infty C \\ \downarrow F & & \downarrow F_w & & \downarrow F_w \\ W(A) & \xrightarrow{w} & \Pi_\infty A & \xleftarrow{f} & \Pi_\infty C \end{array}$$

Then by [Bor11a], the map $f \circ [\phi] : C \rightarrow \Pi_\infty A$ lifts to $W(A)$ as our universal map $g : C \rightarrow W(A)$. It is also clear from the above diagram that $g \circ \phi = F \circ g$. Let us now give an inductive description of the map g . Let $g(x) = (x_0, x_1, \dots) \in W(A)$. Then from the above diagram $w \circ g = f \circ [\phi]$. Therefore for all $n \geq 0$ we have

$$(2.3) \quad x_0^{q^n} + \pi x_1^{q^{n-1}} + \dots + \pi^n x_n = f(\phi^n(x))$$

Note that clearly $x_0 = f(x)$ and $x_1 = f(\delta(x))$. In the case when A has a π -derivation on it, set $C = A$ and $f = \mathbb{1}$ and let us denote the induced universal map by $\exp_\delta := g : A \rightarrow W(A)$ given by $\exp_\delta(x) = (P_0(x), P_1(x), \dots)$. Composing with the restriction map $T : W(A) \rightarrow W_n(A)$ for each $n \geq 0$, we obtain $\exp_\delta : A \rightarrow W_n(A)$.

If A is an R -algebra with $f : R \rightarrow A$ be the structure map, since the identity map $\mathbb{1}$ is a q -power lift of Frobenius on R , we have the universal map $\exp_\delta : R \rightarrow W_n(R)$ for each n . Then $W_n(A)$ is also canonically an R -algebra by the following composition

$$(2.4) \quad R \xrightarrow{\exp_\delta} W_n(R) \xrightarrow{W_n(f)} W_n(A)$$

3. THE ANALOGUE OF TAYLOR EXPANSION OVER WITT VECTORS

We will review the case of differential algebra to motivate the analogue of Taylor expansion in the arithmetic case of Witt vectors. Let $X = \text{Spec } B$ be an affine

smooth curve over \mathbb{C} with a derivation ∂ on it. Let $\mathfrak{p} \in X$ be a closed point on it and let $u : B \rightarrow B_0 := B/\mathfrak{p} = \mathbb{C}$ be the evaluation map at \mathfrak{p} and denote $x(\mathfrak{p}) := u(x)$.

Let $D_n(B) := B[t]/(t^{n+1})$ be the ring of truncated polynomials of length $n+1$ and $D(B) := \varprojlim B[t]/(t^{n+1}) = B[[t]]$. Since B has a derivation ∂ on it, by universal property, consider the Hasse-Schmidt exponential map $\exp_\partial : B \rightarrow D(B)$ given by

$$(3.1) \quad \exp_\partial(x) = \sum_{i=0}^{\infty} \frac{\partial^{(i)}x}{i!} t^i$$

Then note that the Taylor expansion of any function $x \in B$ about \mathfrak{p} and along the derivation ∂ can be realised as the map $\overline{\exp}_\partial : B \rightarrow D(B_0) = \mathbb{C}[[t]]$ given by the following composition

$$(3.2) \quad \begin{array}{ccc} B & \xrightarrow{\exp_\partial} & D(B) \xrightarrow{D(u)} D(B_0) \\ x \mapsto & \sum_{i=0}^{\infty} \frac{\partial^{(i)}x}{i!} t^i \mapsto & \sum_{i=0}^{\infty} \frac{(\partial^{(i)}x)(\mathfrak{p})}{i!} t^i \end{array}$$

Recall that a derivation $\partial : B \rightarrow B$ is a ring homomorphism $B \rightarrow D_1(B)$, the ring of truncated polynomials of length 2. And in the arithmetic case, a π -derivation $\delta : B \rightarrow B$ is in fact a ring homomorphism $B \rightarrow W_1(B)$. The Witt vectors in this arithmetic context plays a similar role as the truncated polynomials in the differential algebra case. Inspired by this analogy, we will now introduce the arithmetic analogue of the Taylor expansion principle.

Let A be a B -algebra which has a lift of the q -power Frobenius ϕ which when restricted to B is the identity. Also consider the evaluation map $u : A \rightarrow A_0 := A/\mathfrak{p}A$ and denote $u(a) = \bar{a}$. Then, by analogy with 3.2 we define the *arithmetic Taylor expansion* of A about \mathfrak{p} and with respect to δ to be $\overline{\exp}_\delta : A \rightarrow W(A_0)$ given by the following composition

$$(3.3) \quad \begin{array}{ccc} A & \xrightarrow{\exp_\delta} & W(A) \xrightarrow{W(u)} W(A_0) \\ \overline{\exp}_\delta(x) = & (\overline{P_0(x)}, \overline{P_1(x)}, \dots) \end{array}$$

Note that $P_0(x) = x$ and $P_1(x) = \delta x$.

4. INJECTIVITY OF THE ARITHMETIC TAYLOR EXPANSION

Let R be a complete discrete valuation ring with maximal ideal $m = (\pi)$ where $\pi \in R$ is a generator and also further assume that the residue field $k = R/(\pi)$ is a finite field of order $q = p^h$ for some h for a fixed prime p . Let A be an R -algebra which is separated and flat over R with a π -derivation δ on it. Also for any $x \in A$, let $v_\pi(x)$ denote the π -adic valuation of x .

For any R -algebra C , define $C_n = C/\pi^{n+1}C$ for all $n \geq 0$. We define the subset of constants, denoted by, $A^\delta := \{a \in A \mid \delta a = 0\}$. For each n , define the subset $T_n := \{x \in W_n(A) \mid x = (x_0, 0, \dots, 0), x_0 \in A\} \subset W_n(A)$. Let $T = \varprojlim T_n$. Then in the case when the $\text{char } A \neq \text{char } k$, T is only closed under multiplication but

not addition. However when A is of equal positive characteristic, T is a subring of $W(A)$ since the Teichmuller map $\theta : A \rightarrow W(A)$ is then a ring homomorphism.

For $n \geq 1$, let $I_n \subset W(A)$ be the ideal defined by

$$I_n := \{x \in W(A_0) \mid x = (x_0, x_1, \dots), x_i \in A_0 \text{ and } x_0 = x_1 = \dots = x_{n-1} = 0\}.$$

Also let $\overline{\text{exp}}_\delta : A \rightarrow W(A_0)$ be as in 3.3, given by $\overline{\text{exp}}_\delta(x) = (\overline{P_0(x)}, \overline{P_1(x)}, \dots)$.

Lemma 4.1. *Let A be as above. If $v_\pi(a) \geq 1$, then $v_\pi(\delta a) = v_\pi(a) - 1$. Moreover if $\delta a = 0$ and $v_\pi(a) \geq 1$ then $a = 0$.*

Proof. If $a = 0$, then there is nothing to show. Let $a = \pi^n b$ where $n = v_\pi(a)$ and hence $\pi \nmid b$ and $n \geq 1$. Then $\delta a = \pi^{n-1}(\phi(b) - \pi^{(q-1)n}b^q) = 0$. But $v_\pi(\phi(b)) = 0$ and hence $v_\pi(\delta a) = v_\pi(a) - 1$.

Since $v_\pi(b) = 0$ implies $v_\pi(\phi(b)) = 0$. If $\delta a = 0$ and $a \neq 0$ then since A is flat over R , that implies $\phi(b) = \pi^{(q-1)n}b^q$ which is a contradiction to $a \neq 0$ and we are done. \square

Lemma 4.2. *Let $n \leq m$. Then the function $H(x) = q^{n-x-1}(m-x) + x - 1$ is a strictly decreasing function in the interval $0 \leq x \leq n-1$.*

Proof. Differentiating H with respect to x we get:

$$H'(x) = -q^{n-1-x}[1 + (\ln q)(m-x)] + 1$$

Note that $1 + (\ln q)(m-x) > 1$ for all $x \leq m-1$ and hence $-q^{n-1-x}[1 + (\ln q)(m-x)] < -1$ which implies $H'(x) < 0$ and we are done. \square

Proposition 4.3. *For all n we have*

$$(4.1) \quad P_n(x) = \sum_{i=0}^{n-1} \sum_{j=1}^{q^{n-1-i}} \pi^{i+j-n} \binom{q^{n-1-i}}{j} P_i(x)^{q^{n-1-i-j}} (\delta P_i(x))^j$$

Proof. From 2.3, we get-

$$(4.2) \quad x^{q^n} + \pi P_1(x)^{q^{n-1}} + \pi^2 P_2(x)^{q^{n-2}} + \dots + \pi^n P_n(x) = \phi^n(x)$$

Since A has a π -derivation we have

$$\begin{aligned} \pi^n P_n(x) + \sum_{i=0}^{n-1} \pi^i P_i(x)^{q^{n-i}} &= \phi(\phi^{n-1}(x)) \\ &= \phi\left(\sum_{i=0}^{n-1} \pi^i P_i(x)^{q^{n-1-i}}\right) \\ &= \sum_{i=0}^{n-1} \pi^i \phi(P_i(x))^{q^{n-1-i}} \\ P_n(x) &= \sum_{i=0}^{n-1} \pi^{i-n} (\phi(P_i(x))^{q^{n-1-i}} - P_i(x)^{q^{n-i}}) \end{aligned}$$

and the result follows by binomially expanding $\phi(P_i(x))^{q^{n-1-i}} = (P_i(x)^q + \pi \delta P_i(x))^{q^{n-1-i}}$, for each i . \square

4.1. The unequal characteristic case. Let $q = p^h = |k|$ as before and $v_\pi(p) = e$ be the absolute ramification index of p in R , that is $pR = (\pi)^e$. Let us define the following

$$(4.3) \quad \begin{aligned} L_{ij} &= \pi^{i+j-n} \binom{q^{n-1-i}}{j} P_i(x)^{q(q^{n-1-i}-j)} (\delta P_i(x))^j \\ S_i &= \sum_{j=1}^{q^{n-1-i}} L_{ij} \end{aligned}$$

Then

$$(4.4) \quad P_n(x) = \sum_{i=0}^{n-1} S_i$$

Lemma 4.4. *If $v_\pi(P_i(x)) = m - i$, $0 \leq i < m$, then $v_\pi(L_{ij}) = i - n + (n - 1 - i)eh + (m - i)q^{n-i} - (m - i)(q - 1)j - v_\pi(j)$.*

Proof. We know that $v_\pi\left(\binom{p^l}{j}\right) = lv_\pi(p) - v_\pi(j)$. Then the result follows easily from the following computation

$$\begin{aligned} v_\pi(L_{ij}) &= i + j - n + (n - 1 - i)eh - v_\pi(j) + (m - i)q(q^{n-1-i} - j) + (m - i - 1)j \\ &= i - n + (n - 1 - i)eh + (m - i)q^{n-i} - (m - i)(q - 1)j - v_\pi(j) \quad \square \end{aligned}$$

Lemma 4.5. *If $v_\pi(P_i(x)) = m - i$, $0 \leq i < m$, then $v_\pi(S_i) = v_\pi(L_{iq^{n-1-i}}) = q^{n-1-i}(m - i) - n + i$.*

Proof. Since A is flat over R , it is sufficient to show that $v_\pi(L_{iq^{n-1-i}}) \leq v_\pi(L_{ij})$ for all $1 \leq j \leq q^{n-1-i} - 1$. Let $C = (m - i)(q - 1) > 0$. Then for all $j, j' = 0, \dots, q^{n-1-i}$ by lemma 4.4 we have $v_\pi(L_{ij'}) - v_\pi(L_{ij}) = C(j - j') + (v_\pi(j) - v_\pi(j'))$

Now for all $0 \leq j \leq q^{n-1-i} - 1$ we have

$$\begin{aligned} C(q^{n-1-i} - j) &> 0 > v_\pi(j) - v_\pi(q^{n-1-i}) \\ 0 &> C(j - q^{n-1-i}) + (v_\pi(j) - v_\pi(q^{n-1-i})) \\ (4.5) \quad &= v_\pi(L_{iq^{n-1-i}}) - v_\pi(L_{ij}) \end{aligned}$$

and we are done. \square

Now we return to the general characteristic case

Theorem 4.6. *Let A be a separated flat R -algebra with a π -derivation δ on it and $x \in A$ with $v_\pi(x) = m < \infty$ (in particular $x \neq 0$). Then for all $n \leq m$, $P_n(x) \neq 0$ and moreover*

$$v_\pi(P_n(x)) = m - n.$$

Proof. We will prove the result by induction on n . For $n = 1$, we have already shown that in lemma 4.1. Assume it is true for $n-1$. Then $v_\pi(P_i(x)) = m - i$ for all $i = 0, \dots, n - 1$.

We wish to first determine the valuation $v_\pi(S_i)$ for all $i = 0, \dots, n - 1$. Let us consider two separate cases of $\text{char } A = p$ and $\text{char } A \neq p$. If $\text{char } A = p$, then

note that $S_i = L_{iq^{n-1-i}}$ and therefore $v_\pi(S_i) = v_\pi(L_{iq^{n-1-i}})$. Otherwise by lemma 4.5 we also have $v_\pi(S_i) = v_\pi(L_{iq^{n-1-i}})$.

Therefore for all $i = 0, \dots, n-1$, we have

$$v_\pi(S_i) = q^{n-1-i}(m-i) - (n-i)$$

By lemma 4.2 the right hand side is a strictly decreasing function of i and hence the minimum is attained at $i = n-1$. Since A is flat, we conclude that

$$\begin{aligned} v_\pi(P_n(x)) &= \min_{i=0, \dots, n-1} (q^{n-1-i}(m-i) + i - n) \\ &= m - n \end{aligned}$$

and we are done. \square

Proof of Theorem 1.1. Suppose $x \in \ker(\overline{\exp}_\delta)$ and $x \neq 0$. Then $\overline{P_i(x)} = 0$ for all $i \geq 0$. Since $\overline{P_0(x)} = \overline{x}$, clearly $v_\pi(x) \geq 1$ and let $n := v_\pi(x) \geq 1$. However, by theorem 4.6, $v_\pi(P_n(x)) = 0$ which implies $\overline{P_n(x)} \neq 0$ and hence is a contradiction and therefore we must have $x = 0$ and we are done. \square

Corollary 4.7. *The induced map $A_n \rightarrow W_n(A_0)$ is an injection.*

Proof. Consider the truncated map $\overline{\exp}_\delta : A \rightarrow W_n(A_0)$ given by $x \mapsto (\overline{P_0(x)}, \dots, \overline{P_n(x)})$. Let x be in the kernel of the map. Then by theorem 4.6, $v_\pi(x) \geq n+1$, that is $x \in \pi^{n+1}A$. And clearly $\pi^{n+1}A$ is contained inside the kernel by the same theorem and hence we are done. \square

Consider the topology on $W(A_0)$ induced by the basis of open sets I_n for all n . Then the above corollary implies the following about the π -adic topology of A :

Corollary 4.8. *Let A be as above. The π -adic topology on A is the restriction of the I_n -adic topology on $W(A_0)$ by the map $\overline{\exp}_\delta$.*

Lemma 4.9. *Let $C \hookrightarrow D$ be two π -adically complete, flat, R -algebras such that $C_0 \simeq D_0$. Then $C \simeq D$.*

Proof. We prove $C_n \simeq D_n$ using induction. Since both C and D are π -torsion free, for all n , we have $\pi^{n-1}C/\pi^n C \simeq C_0 \simeq D_0 \simeq \pi^{n-1}D/\pi^n D$. For $n = 0$, it is the hypothesis. Suppose true for $n-1$. Then

$$\begin{array}{ccccccc} 0 & \longrightarrow & \pi^{n-1}C/\pi^n C & \longrightarrow & C_n & \longrightarrow & C_{n-1} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \pi^{n-1}D/\pi^n D & \longrightarrow & D_n & \longrightarrow & D_{n-1} \longrightarrow 0 \end{array}$$

and we conclude by snake lemma. \square

The following lemma is standard.

Lemma 4.10. *Let B, π and k be as above. Also further assume B is a k -algebra. Then $R \simeq k[[\pi]]$.*

Proof. Since B is a k -algebra implies R is a k -algebra. We claim that the image of $k[[\pi]]$ in R is injective. Let $f \in k[[\pi]]$ be such that its image in R is 0. Then there exists an $l \geq 0$ such that f can be written as $f = \sum_{i=0}^{\infty} \alpha_{l+i} \pi^{l+i}$, where $\alpha_{l+i} \in k$

for all i and $a_l \neq 0$. Since R is π -torsion free, that implies $\alpha_l = \pi(\alpha_{l+1} + \dots)$ which implies $v_\pi(\alpha_l) \geq 1$. But since $\alpha_l \in k$ we must have $\alpha_l = 0$ which is a contradiction and this proves the claim. Now we have $k[[\pi]] \subseteq R$ and both are π -adically complete and has the same residue field k . This implies they are isomorphic by lemma 4.9 and we are done. \square

If A is of equal positive characteristic, then since the Witt vector addition is component wise linear, it makes T a subgroup under addition too and hence is a subring of $W(A_0)$. Therefore A^δ , which is isomorphic to T , is also a subring of A .

Lemma 4.11. *If $\delta x = 0$ then $P_n(x) = 0$ for all $n \geq 1$.*

Proof. We will prove by induction on n . It is true for $n = 1$ by hypothesis. If true for $n - 1$ then by proposition 4.3, $P_n(x)$ can be expressed as

$$P_n(x) = \sum_{i=0}^{n-1} \sum_{j=1}^{q^{n-1-i}} \pi^{i+j-n} \binom{q^{n-1-i}}{j} P_i(x)^{q^{n-1-i}-j} (\delta P_i(x))^j$$

which clearly implies $P_n(x) = 0$ and we are done. \square

Lemma 4.12. *For all $1 \leq x \leq n - 1 \leq m$, $H(x) = x - n + q^{n-1-x}(m - x + 1)$ is a strictly decreasing function in x .*

Proof. Differentiating H we get

$$H'(x) = 1 - q^{n-1-x}((m - x + 1) \ln q + 1)$$

Note that $q^{n-1-x} \geq 1$ and $(m - x + 1) \ln q + 1 > 1$ for all $1 \leq x \leq n - 1 \leq m$ and therefore $H'(x) < 0$ and we are done. \square

Proposition 4.13. *Let R be of equal characteristic and $x \in A$ be such that $v_\pi(x) = 0$ and $v_\pi(\delta x) = m \geq 1$. Then $v_\pi(P_n(x)) = m - n + 1$ for all $1 \leq n \leq m + 1$.*

Proof. We have for all n ,

$$P_n(x) = \sum_{i=0}^{n-1} S_i$$

where $S_i = \pi^{i-n}(\phi(P_i(x))^{q^{n-1-i}} - P_i(x)^{q^{n-1-i}})$ for all $i = 0 \dots n - 1$. We will prove our result by induction. For $n = 1$, $P_1(x) = \delta x$ and hence the result is true by hypothesis. Now assume true for $n - 1$. Since $\text{char } R > 0$ we have $S_0 = \phi(x)^{q^{n-1}} - x^{q^n} = (x^{q^n} + (\pi \delta x)^{q^{n-1}}) - x^{q^n} = (\pi \delta x)^{q^{n-1}}$ which implies, $v_\pi(S_0) = q^{n-1}(1 + m)$.

Now for all $1 \leq i \leq n - 1 \leq m$, we have $v_\pi(\phi(P_i(x))^{q^{n-1-i}}) < v_\pi(P_i(x)^{q^{n-1-i}})$ and therefore we have

$$v_\pi(S_i) = i - n + q^{n-1-i}(m - i + 1)$$

Then by lemma 4.12, the valuation $v_\pi(S_i)$ is a strictly decreasing function of i and therefore we have

$$v_\pi\left(\sum_{i=1}^{n-1} S_i\right) = v_\pi(S_{n-1}) = m - n + 1$$

Note that $v_\pi(S_0) = q^{n-1}(1+m) > m-n+1 = v_\pi(\sum_{i=1}^{n-1} S_i)$ for all $n > 1$. Therefore we have

$$v_\pi(P_n(x)) = v_\pi\left(\sum_{i=0}^{n-1} S_i\right) = v_\pi\left(\sum_{i=1}^{n-1} S_i\right) = m - n + 1$$

and we are done. \square

Proposition 4.14. *If R is of positive characteristic, then $\overline{\exp}_\delta(A^\delta) = \overline{\exp}_\delta(A) \cap T$.*

Proof. Let $x \in A^\delta$. Then since $\delta x = 0$, by lemma 4.11, we have $P_n(x) = 0$ for all $n \geq 1$ and hence $\overline{\exp}_\delta(A^\delta) \subseteq \overline{\exp}_\delta(A) \cap T$.

Conversely, consider an element $y \in \overline{\exp}_\delta(A) \cap T$. Then there exists an $x \in A$ such that $\overline{\exp}_\delta(x) = y$ and satisfies $\overline{P}_n(x) = 0$ for all $n \geq 1$. Since $\overline{\exp}_\delta$ is injective, it is enough to consider the case when $\overline{x} \neq 0$. Now if $\delta x = 0$ then we are done. Otherwise if $\delta x \neq 0$ then we have $0 < v_\pi(\delta x) = m < \infty$, since $\overline{P}_1(x) = \overline{\delta x} = 0$ and A is separated. But by proposition 4.13 we have $v_\pi(P_{m+1}(x)) = 0$ and hence we have $\overline{P}_{m+1}(x) \neq 0$ which is a contradiction and therefore we must have $\delta x = 0$ and therefore $\overline{\exp}_\delta(A) \cap T \subseteq \overline{\exp}_\delta(A^\delta)$ and we are done. \square

Lemma 4.15. *If R is of positive characteristic, then $A^\delta \cap \pi A = (0)$. In particular the composition*

$$A^\delta \hookrightarrow A \xrightarrow{u} A_0$$

is an injection.

Proof. This follows because $\overline{\exp}_\delta(A^\delta) \subseteq T$ and by proposition 4.14, $T \cap \overline{\exp}_\delta(\pi A) = (0)$ as well as the map $\overline{\exp}_\delta$ is an injection. \square

Lemma 4.16. *Let R be of positive characteristic and A be a separated flat R -algebra. Then the subring generated by finite sums of A^δ and π inside A is $A^\delta[\pi]$.*

Proof. If not there exists $0 \neq f = a_l \pi^l + \dots + a_n \pi^n$ where $a_l, \dots, a_n \in A^\delta$ and $a_l \neq 0$ such that $f = 0$. Since A is flat over R , we have $a_l = -a_{l+1} \pi + \dots + a_n \pi^{n-l} \in \pi A$, that is, $v(a_l) \geq 1$. But since $a_l \in A^\delta$ we also have $\delta a_l = 0$. But that implies, by lemma 4.1, $a_l = 0$ which is a contradiction. Therefore we must have $f = 0$ and we are done. \square

Corollary 4.17. *Let R be of equal characteristic and A be a separated flat π -adically complete R -algebra. Then the π -adic completion of the subring generated by A^δ and π inside A is $A^\delta[[\pi]]$.*

Proof of Theorem 1.2. Since ϕ induces an isomorphism on A_0 , we have that $\overline{\exp}_\delta : A \simeq W(A_0)$ is an isomorphism. Therefore by proposition 4.14 we have $A^\delta \simeq T \simeq A_0$. Hence $A^\delta[[\pi]] \hookrightarrow A$ and the result follows from lemma 4.9. \square

Corollary 4.18. *Let A be a π -torsion free separated and π -adically complete R -algebra with a lift of Frobenius ϕ that induces an isomorphism on A_0 . Also further assume B is a k -algebra. Then $A_n \simeq A^\delta \otimes_k B_n$.*

Proof. From theorem 1.2 we have $A_n \simeq A^\delta \otimes_k k[\pi]/(\pi^{n+1})$. By lemma 4.10, we have $B_n \simeq k[\pi]/(\pi^{n+1})$ and we are done. \square

5. AN APPLICATION IN THE CASE OF SCHEMES

Let (B, \mathfrak{p}) be as before. Furthermore assume B is also a k -algebra where $k = B/\mathfrak{p}$. And as before, let R be the completion of B with respect to \mathfrak{p} and $\iota : B \rightarrow R$ be the canonical injective map. Let A be a B -algebra and let \hat{A} denote the completion of A with respect to $\mathfrak{p}A$. Then \hat{A} is an R -algebra.

Lemma 5.1. *Let A be π -torsion free, then \hat{A} is π -torsion free and separated.*

Proof. Since A has no π -torsion, \hat{A} is π -torsion free as well. And \hat{A} is separated since it is the π -adic completion of A . \square

Proof of Theorem 1.3. Let $S_n = \operatorname{Spec} B/\mathfrak{p}^{n+1}$. For any scheme Z over S , let $Z_n := Z \times_S S_n$. Consider $Y = X_0 \times_{\operatorname{Spec} k} S$. Since X is an integral scheme of finite type over S , by lemma 5.1, we have that $\mathcal{O}_{\hat{X}}$ is a sheaf of π -torsion free separated Noetherian R -algebras. Therefore by corollary 4.18, we have for each n , compatible isomorphisms $f_n : X_n \rightarrow Y_n$. Then by Artin approximation, [Artin69] cor 2.4, there exists an étale neighbourhood S' of $\mathfrak{p} \in S$ such that if $X' = X \times_S S'$ then $X' \simeq Y \times_S S' \simeq X_0 \times_{\operatorname{Spec} k} S'$. \square

REFERENCES

- [Artin69] M. F. Artin. Algebraic approximation of structures over complete local rings. *Publications Mathématiques de l'IHÉS*, 36:23–58, 1969.
- [Bor11a] J. Borger. The basic geometry of Witt vectors, I: The affine case. *Algebra and Number Theory*, 5(2):231–285, 2011.
- [Bui87] A. Buium. Fields of definition of algebraic varieties in characteristic zero. *Compositio Mathematica*, 61(3):339–352, 1987.
- [Bui95] A. Buium. Differential characters of abelian varieties over p -adic fields. *Inventiones mathematicae*, 122(1):309–340, 1995.
- [Bui00] A. Buium. Differential modular forms. *Journal für die Reine und Angewandte Mathematik*, 520:95–168, 2000.
- [Bui09] A. Buium and B. Poonen. Independence of points on elliptic curves arising from special points on modular and shimura curves, II: local results. *Compositio Mathematica*, 145(03):566–602, 2009.
- [G02] H. Gillet. Differential algebra- a scheme theory approach. *Differential algebra and related topics (Newark, NJ, 2000)*, pages 95–123, 2002.
- [H05] L. Hesselholt. Lecture notes on witt vectors. *Survey article, unpublished*, 2005.
- [J85] André Joyal. δ -anneaux et vecteurs de witt. *CR Math. Rep. Acad. Sci. Canada*, 7(3):177–182, 1985.